Online Supplemental Material: Deriving Formula for Approximating the Variance

Below we derive the formula for approximating the variance $VAR(\bar{Y}/\bar{X})$. We let $g: (\hat{u}, \widehat{w}) \rightarrow \hat{u}/\widehat{w}$ be a 2-to-1 one mapping that transforms a pair of estimators $\hat{u}$ and $\widehat{w}$ (both are random variables) into a ratio. Assume that the first-order partial derivatives of $g(\hat{u}, \widehat{w})$ exist. The delta method[32, 33] shows that the variance of $\hat{u}/\widehat{w}$ can be approximated as (the $\cong$ sign indicates an approximation):

$$VAR\left(\frac{\hat{u}}{\widehat{w}}\right) = VAR\big(g(\hat{u}, \widehat{w})\big) \cong G \cdot V \cdot G^T \text{ , where}$$

- $G = \left[\frac{\partial g}{\partial \hat{u}}, \frac{\partial g}{\partial \widehat{w}}\right]$ is a vector of partial derivatives,
- G^T is the transpose of matrix G, and
- V is the estimated variance-covariance matrix of $(\hat{u}, \widehat{w})$.

The power rule for derivatives indicates that

$$\frac{\partial g}{\partial \hat{u}} = \frac{1}{\widehat{w}} \quad ; \quad \frac{\partial g}{\partial \widehat{w}} = \frac{-\hat{u}}{\widehat{w}^2}$$

Thus,

$$G \cdot V \cdot G^T = \left[\frac{1}{\widehat{w}}, \frac{-\hat{u}}{\widehat{w}^2}\right] \cdot \begin{bmatrix} VAR(\hat{u}) & COV(\hat{u}, \widehat{w}) \\ COV(\hat{u}, \widehat{w}) & VAR(\widehat{w}) \end{bmatrix} \cdot \left[\frac{1}{\widehat{w}}, \frac{-\hat{u}}{\widehat{w}^2}\right]^T$$

In the special case where $\hat{u}$ and $\widehat{w}$ are estimators from two independent surveys, their covariance is zero. Substituting $\hat{u}$ with $\bar{Y}$ (e.g., sample mean of the survey for the numerator, the NHIS) and $\widehat{w}$ with $\bar{X}$ (e.g., sample mean of the survey for the denominator, the ATUS), we obtain

$$VAR\left(\frac{\bar{Y}}{\bar{X}}\right) \cong \left[\frac{1}{\bar{X}}, \frac{-\bar{Y}}{\bar{X}^2}\right] \cdot \begin{bmatrix} VAR(\bar{Y}) & 0 \\ 0 & VAR(\bar{X}) \end{bmatrix} \cdot \left[\frac{1}{\bar{X}}, \frac{-\bar{Y}}{\bar{X}^2}\right]^T = \frac{VAR(\bar{Y})}{\bar{X}^2} + \frac{\bar{Y}^2 \cdot VAR(\bar{X})}{\bar{X}^4} \quad \ldots \ldots \, equation\ (2)$$